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An Estimation of the Fine Structure Constant Using Fiber Bundles

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We calculate g_0/e where g_0 is the strength of an elementary magnetic monopole and e the charge on the electron, in terms of a ratio of loop sizes in the twisted and untwisted principal fiber bundles with $U(1)$ the structure group and $R^3-\{0\}$ the base space. The result involves the present distance around the $U(1)$ space and, rather surprisingly, the structure of the quantum gravitational vacuum. Combining our result with the expression for eg_0 from the Dirac quantization condition gives a final estimate for the fine structure constant, $\alpha \sim 1/100$.

1. INTRODUCTION

Several attempts have been made to calculate the electromagnetic fine structure constant, α. The long program of Johnson et al. (1964, 1967, 1973), of Baker et al. (1969, 1971), and of Adler (1972) to calculate α as the value of the UV-stable fixed point in quantum electrodynamics ran into computational and perhaps conceptual difficulties (Gross and Wilczek, 1973). Wyler (1969, 1971) considered the seven-dimensional group O(5, 2) and equated α to the volume of the full group divided by the volume of the subgroup of the five real dimensions. The resulting number is very close to the experimental value for α, but the physical relevance of these groups is not at all clear.

In the present paper, we present the idea that α is related to the geometrical structure of a principal fiber bundle with $U(1)$ as the structure group and $R^3-\{0\}$ (corresponding to a static charge) as the base space. It is well known that an electric charge e corresponds to the untwisted trivial bundle while a magnetic monopole g corresponds to the twisted bundle.

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Since magnetic monopoles are fundamentally involved, we can use the Dirac (1931, 1948) quantization condition

$$eg/\hbar c = n/2 \tag{1}$$

where n is the number of fundamental magnetic monopole charges present, to relate e and g. g_0 will be used in the following to refer to the $n=1$ case in (1) so that in general $g = ng_0$. We will relate g_0/e to the structure of the relevant fiber bundles and then combine our result with Dirac's to estimate e and g_0 separately.

We present the details below. In Section 2, we look at the structure of the fiber bundles in detail and find an expression for g_0/e in terms of loops in the $U(1)$ space and in space-time. We then discuss the quantities we need in this expression, calculate the fine structure constant α, and make a few concluding remarks in Section 3.

2. STRUCTURE OF THE FIBER BUNDLE

We want to find an expression for g_0/e in terms of the structure of the relevant fiber bundles. We consider a principal fiber bundle E with $U(1)$ as both the structure group G and typical fiber F (Daniel and Viallet, 1980). Since we are interested in a static charge, we take the base space B to be $R^3 - \{0\}$ where the position of the charge (or magnetic monopole) at the origin is deleted, exactly as in the work of Wu and Yang (1975) on magnetic monopoles. The topological properties of a fiber bundle are unmodified if the base is contracted. Since S^2 is the pullback of $R^3 - \{0\}$, we can take S^2 to be our base space. Thus both electric charges and magnetic monopoles are represented by a $U(1)$ (homeomorphic to S^1) principal bundle over S^2.

Let us define the fiber bundle more exactly using the work of Choquet-Bruhat, DeWitt-Morette, and Dillard-Bleicke (1977). Let Π be a continuous surjective mapping Π: $E \to B$. Then $\Pi^{-1}(x)$ is called the fiber at x, also denoted by F_x, where $x \in B$. Let B be covered by a family of open sets $\{U_j : j \in J \subseteq N\}$. Then a fiber bundle must satisfy the following:

(1) Locally the bundle is homeomorphic to a product bundle. Thus $\Pi^{-1}(U_j)$ is homeomorphic to $U_j \times F$ for all $j \in J$. The homeomorphism $\phi_j : \Pi^{-1}(U_j) \to U_j \times F$ has the form $\phi_j(P) = (\Pi(P), \overset{\Delta}{\phi}_j(P))$. Thus $\overset{\Delta}{\phi}_j|_{F_x}$ also denoted by $\overset{\Delta}{\phi}_{j,x}$ is a homeomorphism from F_x onto F.

(2) The structure of the fiber bundle is determined by what happens in the overlap region. Let $x \in U_j \cap U_k$. The homeomorphism $\overset{\Delta}{\phi}_{k,x} \circ \overset{\Delta}{\phi}{}_{j,x}^{-1}$: $F \to F$ is an element of the structural group G for all j, $k \in J$. If G has only one element the bundle is trivial.

(3) The induced mapping g_{jk}: $U_j \cap U_k \to G$ by $x \to g_{jk}(x) = \overset{\Delta}{\phi}_{k,x} \circ \overset{\Delta}{\phi}{}^{-1}_{j,x}$ is continuous. $g_{jk}(x)$ is known as the transition function.

Now for the present case, in general, two coordinate patches are required to cover S_2. These can be chosen to be

$$\begin{aligned} U_1&: 0 \leq \theta < \pi/2 + \delta, \qquad 0 \leq \phi < 2\pi \\ U_2&: \pi/2 - \delta < \theta \leq \pi, \qquad 0 \leq \phi < 2\pi \end{aligned} \tag{2}$$

with $0 < \delta \leq \pi/2$. These overlap in a band along the equator. Following Wu and Yang (1975), if a magnetic monopole is present, a singularity-free vector potential (connection) can be separately written down in regions U_1 and U_2. In the overlap region, these two vector potentials are related by a gauge transformation $\exp(2ige\phi/\hbar c)$. Requiring this gauge transformation to be single valued leads directly to the quantization condition (1). In fiber bundle language the transition function is

$$g_{12}(\phi) = \exp(2ige\phi/\hbar c) = \exp(in\phi) \tag{3}$$

where e is the electron charge and g the charge of a magnetic monopole. This gives the structure of the fiber bundle. We note that the overlap region above can be contracted to S^1 (the equator of S^2). The transition function (3) maps this S^1 overlap region in the base space (as ϕ ranges 0 to 2π) into the structural group which is $U(1)$ (homeomorphic to S^1) in the present case. Clearly (3) is a one-dimensional unitary transformation for any value of ϕ.

The integer n tells how many times the overlap region is wrapped around the $U(1)$ space. Thus if $n = 0$ in (3) so that no magnetic monopoles are present, we have $g_{12}(\phi) = 1$ and thus a trivial product bundle. In this case, the overlap region S^1 is mapped into a single point of $U(1)$ and we have an electric charge present. If $n = 1$, the overlap region S^1 is mapped around $U(1)$ (or S^1) exactly once, and we have a nontrivial bundle. A particle with one magnetic charge is present. If $n = 2$, the overlap region S^1 is mapped around $U(1)$ twice. A particle with magnetic charge of twice the fundamental unit is present, and so on. To summarize: If S^1 maps to a point of $U(1)$, an electric charge is present; if S^1 maps around the $U(1)$ space n times, then a magnetic monopole of charge ng_0 is present.

From the above discussion, *the only difference between an electric charge and an elementary magnetic monopole g_0 in the fiber bundle formalism is whether the overlap region S^1 maps to a point of $U(1)$ or wraps around it once.* Now how can we relate this mathematical fact to the strength of the gauge coupling α? The work of Souriau (1963) and of Chodos and Detweiler (1980) provides the clue. Both groups find that if one considers the Klein-Gordon equation in Kaluza-Klein (1921, 1926) space (or equivalently in our fiber bundle which is a five-dimensional Riemannian manifold complete with metric) and assumes that the wave function of the particle is periodic with respect to the fifth coordinate, then in four dimensions the particle satisfies the usual Klein–Gordon equation with minimal coupling to the electromagnetic field and with charge

$$e=(\hbar c)^{1/2}\frac{4\pi L_P}{2\pi R} \tag{4}$$

$L_P \equiv (\hbar G/c^3)^{1/2}$ is the Planck length and R the radius of the compact $U(1)$ space. Note that the distance around the $U(1)$ space cannot be calculated in the above Kaluza–Klein type theories because these theories are globally scale invariant (Gross and Perry, 1983). Attempts to calculate this distance around the $U(1)$ space by using the breaking of dilatation invariance due to quantum effects have also failed (Appelquist and Chodos, 1983). The mass of the dilation is not determined. (4) does not in any way restrict or determine e but merely relates it to the unknown R.

Now (4) tells us that (I) the electric charge e is related to the circumference $2\pi R$ of the compact space $U(1)$ in the Kaluza-Klein or fiber bundle model context. [This relationship to e makes sense since Bergmann (1942) has shown that the circumference of a closed space, in which all self-intersecting geodesics are closed lines without discontinuities of direction, is a characteristic constant of that space.] We also have the fact found above that (II) the only difference between an electric charge and an elementary magnetic monopole is whether the overlap region S^1 maps to a point of $U(1)$ or wraps around it once. (I) and (II) combined then naturally lead to the following hypothesis:

$$\frac{g_0}{e}=\frac{\text{circumference of the compact space } U(1)}{\text{“distance around a point”}} \tag{5}$$

We need to refine what we mean by the expression (5). Both the numerator and denominator must be taken to refer to the respective distances in physical terms. (4) gives us the numerator directly. The "distance around a point" in the denominator must refer not to a mathematical point but to the distance around the tiny black hole that makes the dominant contribution to the quantum gravitational vacuum (Hawking, 1978). We shrink our overlap region S^1 as far as quantum gravity will allow to get this "distance around a point". Note that we really want a ratio of distances around loops in (5) because magnetic monopoles are characterized by $\Pi_1(U_1)$ and thus homotopies of loops are the relevant mathematical structures. Thus a more precise way to write (5) is in the form

$$\frac{g_0}{e}=\frac{\text{circumference of the compact space } U(1)}{\text{circumference of the dominant space-time loop of quantum gravity}} \tag{6}$$

3. ESTIMATION OF THE FINE STRUCTURE CONSTANT AND DISCUSSION

To complete our calculation of g_0/e in (6) we need an expression for the denominator. For this we need quantum gravity, since that determines the Planck-scale structure of space-time. Unfortunately, no complete, self-consistent theory of quantum gravity exists, so that only an estimate is possible. Hawking (1978) has looked at the quantum gravitational vacuum and has estimated that the dominant contribution to the path integrals is roughly Planck mass black holes with about one black hole per Planck volume. Following Hawking (1978), to estimate the mass M of the black holes giving the dominant contribution, one must maximize the partition function $We^{-\hat{I}}$ as a function of M. Using $W = M/2$, from Gibbons and Hawking (1977) and the black hole action $\hat{I} = 4\pi M^2$ gives $M_{\text{dominant}} = 1/(8\pi)^{1/2}$ in units of the Planck mass. The distance around a black hole is $4\pi M$ so that the

$$\text{circumference of the dominant space-time loop} \approx \frac{4\pi}{(8\pi)^{1/2}} L_P \tag{7}$$

where L_P is the Planck length.

We are now ready to plug the circumference of the compact space $U(1)$ from (4) and the circumference of the dominant space-time loop from (7) into (6) and thus to calculate g_0/e. We find

$$\frac{g_0}{e} = \frac{(8\pi)^{1/2}}{\sqrt{\alpha}} \tag{8}$$

$\alpha \equiv e^2/\hbar c$, however, so (8) gives

$$g_0 = (8\pi)^{1/2}(\hbar c)^{1/2} \tag{9}$$

directly. Putting this into the Dirac quantization condition (1) with $n = 1$, $eg_0/\hbar c = 1/2$, then gives

$$\alpha = \frac{1}{32\pi} \sim \frac{1}{100} \tag{10}$$

for the fine structure constant. Since we considered a static charge, this represents the low energy α. Also since the dressed, physical α is the one responsible for the size of the $U(1)$ dimension (Chodos and Detweiler, 1980), this result would represent the α measured in the laboratory.

The result (10) comes out surprisingly close to the experimental value of $1/137$, especially considering the uncertainties in the calculation. One might hope that a more refined treatment of the fiber bundles along the lines suggested here with a better treatment of the quantum gravitational vacuum and of the compactification of the $U(1)$ space would lead eventually to agreement with experiment. This paper should be considered only a first small step. The biggest surprise in our calculation is that quantum gravity is involved in the calculation of α at all. This is consistent with the work of Salam and Strathdee (1970) on the role that quantum gravity might play in regulating infinities in QED.

Our method of calculating g_0/e from the geometry of the relevant twisted and untwisted fiber bundles has a certain conceptual attractiveness, in spite of the above uncertainties. Extending the above method to the calculation of the coupling constants of non-Abelian gauge theories should be possible, but will undoubtedly hold a few surprises.

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